Exercise Sheet 3: Co-H-Spaces

Tyrone Cutler

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1 Co-H-Spaces

For this exercise sheet we will return to the pointed category. In particular *all spaces, maps and homotopies will be based.* We will introduce co-H-spaces and study their basic properties. These spaces are dual to the so-called H-spaces which were originally defined by Serre [3] to be the homotopical analogues of topological groups. *You are asked to complete all three exercises in section 2 and three of the exercises in section 3.*

Definition 1 *A* ***co-H-Space*** *is a pair* (X, c) *consisting of a pointed space* X *and a map* $c : X \to X \vee X$ *which makes the next diagram commute up to homotopy*

$$\begin{tikzcd} & X \vee X \arrow[d, hook] \\ X \arrow[ur, "c"] \arrow[r, "\Delta"'] & X \times X \end{tikzcd} \tag{1.1}$$

where $\Delta : x \mapsto (x, x)$ *is the diagonal.* $\square$

The map c is called a **comultiplication**. Note that it is an important part of the co-H-structure, and although we will normally denote a co-H-space simply as X, the particular choice of comultiplication must be remembered: the same space may admit many non-equivalent comultiplications. Notice, however, that the condition for c to be a comultiplication depends only on its homotopy class.

Exercise 1.1 Show that the map $c : S^1 \to S^1 \vee S^1$ defined in equation (2.2) of *The Fundamental Group* is a comultiplication. $\square$

Exercise 1.2 Let X be a space. Show that if (Y, c) is a co-H-space, then $X \wedge Y$ is a co-H-space. Conclude that the suspension ΣX is a co-H-space. Explicitly write down a comultiplication for it. $\square$

We call the comultiplication constructed here the **suspension comultiplication** on ΣX. The exercise shows that we have already encountered many co-H-spaces. In particular all spheres S^n for $n \geq 1$ are co-H-spaces. On the other hand, not all co-H-spaces are suspensions. Moreover, there are suspensions which admit comultiplications which are not homotopic to the suspension comultiplication. These examples are rather subtle, however, and we will need to develop some powerful tools to be able to understand them.

Example 1.1 *The wedge inclusions induce an isomorphism $\widetilde{H}^*(X \vee X) \cong \widetilde{H}^*X \oplus \widetilde{H}^*X$. The inverse is induced by the pinch maps $q_1, q_2 : X \vee X \to X$. In particular, if c is a comultiplication on X, then*

$$\begin{tikzcd}
 & \widetilde{H}^*X \oplus \widetilde{H}^*X \arrow[dl, "q_1^* \oplus q_2^*"', "\cong"] \arrow[dr, "+"] & \\
\widetilde{H}^*(X \vee X) \arrow[rr, "c^*"] & & \widetilde{H}^*X
\end{tikzcd} \tag{1.2}$$

commutes. $\square$

Where does the Definition 1 come from? Well, a topological magma is a pair of a space G and a map $m : G \times G \to G$ which makes the following triangle *commute strictly*

$$\begin{tikzcd}
G \vee G \arrow[d, hook] \arrow[dr, "\nabla"] & \\
G \times G \arrow[r, "m"'] & G.
\end{tikzcd} \tag{1.3}$$

The map m describes a product, and the diagram is the requirement that the basepoint of G be a strict unit for it. If we ask only for the diagram to commute up to homotopy we come to the definition of an *H-space*. Thus these objects are direct generalisations of familiar objects such as topological groups and Lie groups.

We get from here to Definition 1 simply by reversing *all* the arrows in (1.3). Categorically this is sensible, since diagrammatically it is no more difficult to ask for the existence of a comagma than for a magma. However, the notion of a co-H-space is something new to the homotopy category. As it turns out, there is no interesting analogue for these objects in Top_*.

Exercise 1.3 Replace the requirement that the square (1.1) commute up to homotopy by asking instead that it commute strictly. Show that there is only a single space satisfying this new definition. $\square$

Fix a co-H-space (X, c). For any space Y and maps $f, g : X \to Y$ define $f + g : X \to Y$ to be the composite

$$f + g : X \xrightarrow{c} X \vee X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\nabla} Y. \tag{1.4}$$

Exercise 1.4 Show that the homotopy class of $f+g$ depends only on the homotopy classes of the maps f, g. Conclude that for any space Y there is a pairing

$$[X,Y] \times [X,Y] \to [X,Y] \tag{1.5}$$

which turns the homotopy set $[X,Y]$ into a magma[1]. What is the unit? Is the product associative? Show that if $\alpha : Y \to Y'$ is a pointed map, then the induced map

$$\alpha_* : [X,Y] \to [X,Y'], \qquad f \mapsto \alpha \circ f \tag{1.6}$$

is a morphism of unital magmas. □

Example 1.2 *Using the observations in example 1.3 we see easily that as maps $\widetilde{H}^*Y \to \widetilde{H}^*X$ the relation*

$$(f+g)^* = f^* + g^*. \tag{1.7}$$

holds. □

The exercise demonstrates one of the main reasons why co-H-spaces are useful to us. It is generally very difficult to compute homotopy sets, and any kind of structure on them is very useful. Of course, it would be desirable to refine this structure if possible.

Definition 2 *Let (X,c) be a co-H-space. Then X is said to be **coassociative** if the following diagram commutes up to homotopy*

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "c"'] & X \vee X \arrow[d, "1\vee c"] \\ X \vee X \arrow[r, "c\vee 1"] & X \vee X \vee X. \end{tikzcd} \tag{1.8}$$

*The co-H-space X is said to be **cocommutative** if the following diagram commutes up to homotopy*

$$\begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "c"] & \\ X \vee X \arrow[rr, "T"] & & X \vee X \end{tikzcd} \tag{1.9}$$

where $T : X \vee X \to X \vee X$ is the twist map which interchanges the factors. □

Definition 3 *Let (X,c) be a co-H-space. A homotopy **coinverse** for X is a map $\iota : X \to X$ making both the following diagrams commute up to homotopy*

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[dr, "*"'] & X \vee X \arrow[d, "(1,\iota)"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "c"] \arrow[dr, "*"'] & X \vee X \arrow[d, "(\iota,1)"] \\ & X. \end{tikzcd} \tag{1.10}$$

□

[1]Some terminology. A **magma** was defined above in the topological case. It is a set M with a binary operation $(x,y) \mapsto x+y$. In particular it need not be associative or commutative. A unit is an element $0 \in M$ such that $0 + x = x + 0$, $\forall x \in M$. An associative magma is called a **semigroup**. An associative magma with unit is called a **monoid**. A monoid with inverses is a group. A morphism of (unital) magmas $M \to N$ is a map preserving binary operations (and units).

A coassociative co-H-space with coinverse is said to be **grouplike**, or is simply called a **cogroup**.

Exercise 1.5 Show that if X is a grouplike co-H-space and Y is any space, then $[X, Y]$ is a group, and is abelian if X is cocommutative. Conclude that the functor

$$hTop_* \xrightarrow{[X,-]} Set_* \tag{1.11}$$

actually takes values in the category *Gro* of groups, and in the category *Ab* of abelian groups if X is cocommutative. □

Thus if X is a cogroup, then there is a functor $\mathcal{G}_X$ filling in the dotted arrow below

$$\begin{tikzcd} & Gro \arrow[d, "forget"] \\ hTop \arrow[ur, dashed, "\mathcal{G}_X"] \arrow[r, "{[X,-]}"'] & Set_*. \end{tikzcd} \tag{1.12}$$

We stress that the lift $\mathcal{G}_X$ depends on the particular choice of cogroup structure on X and need not be unique. Potentially there could even be many such functors $\mathcal{G}_X$, even for a fixed cogroup structure on X.

Definition 4 *A* ***covariant group operation*** *induced by a space X is a lift $hTop_* \xrightarrow{\mathcal{G}_X} Gro$ as in* (1.12) *such that for each space Y, the zero map $* \in \mathcal{G}_X(Y) = [X, Y]$ is the identity.* □

Exercise 1.6 Show that exercise 1.5 has a converse. In detail, show that if a space X induces a covariant group operation $hTop_* \xrightarrow{\mathcal{G}_X} Gro$, then X is a grouplike co-H-space. Explicitly write down a homotopy class of comultiplication on X. □

Putting the conclusions of exercises 1.5 and 1.6 together we find that you have proved the following.

Proposition 1.1 *There is a one-to-one correspondence between the set of homotopy classes of cogroup structures on X and the set of covariant group operations induced by X.* ∎

Of course, similar definitions and statements could be formulated for the more general case of co-H-structures, or for the case of, say, cocommutative cogroup structures. We'll leave the reader to spell these details out on their own.

Definition 5 *Let (X, c_X), (Y, c_Y) be co-H-spaces. A map $f : X \to Y$ is said to be a* ***co-H-map*** *if the next diagram commutes up to homotopy*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "c_X"'] & Y \arrow[d, "c_Y"] \\ X \vee X \arrow[r, "f \vee f"] & Y \vee Y \end{tikzcd} \tag{1.13}$$

□

Exercise 1.7 Show that if $f : X \to X'$ is a co-H-map between grouplike co-H-spaces, then $f^* : [X', Y] \to [X, Y]$ is a group homomorphism. □

Example 1.3 *If $f : X \to Y$ is any map, then $\Sigma f : \Sigma X \to \Sigma Y$ is a co-H-map.* □

2 Homotopy Groups

In the lectures I defined the fundamental group $\pi_1 Y$ of a space Y. There was mention of higher homotopy groups $\pi_n Y$ for each $n \geq 2$, but no definition appeared. It is the purpose of this section to rectify this. Since the basic idea can be generalised, we will focus on the more general case of studying homotopy sets of the form $[\Sigma X, Y]$.

Exercise 2.1 Show that if X is any space, then the suspension comultiplication on ΣX is coassociative and has a coinverse $\iota_X : \Sigma X \to \Sigma X$. Write down explicit homotopies to show that the double suspension $\Sigma^2 X$ is cocommutative, and that the relation $\iota_{\Sigma X} \simeq \Sigma \iota_X \simeq \tau$ holds, where $\tau(x \wedge s \wedge t) = x \wedge t \wedge s$. $\square$

According to this exercise S^n for $n \geq 2$ is a cocommutive cogroup. Thus by the obvious extension of Proposition 1.1 we see that $[S^n, X]$ is an abelian group for each space X, and that this structure is covariantly functorial.

Definition 6 *For a pointed space X and an integer $n \geq 2$ we define its n^{th}* ***homotopy group*** *to be the set $\pi_n X = [S^n, X]$ given the abelian group structure determined by the suspension cogroup structure on S^n.* $\blacksquare$

In particular we have functors

$$hTop_* \xrightarrow{\pi_n} Ab, \qquad n \geq 2. \tag{2.1}$$

We will show in a later lecture using the *Cellular Approximation Theorem* that the covariant group structures on these functors are unique. In fact we will show that S^n for $n \geq 2$ has a unique co-H-structure. The case $n = 1$, on the other hand, is distinct. So exactly how unique is the group structure on $\pi_1 X$?

Exercise 2.2 Using the Seifert-van Kampen Theorem we can show that

$$\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} \tag{2.2}$$

is a free group generated by the two inclusions $in_1, in_2 : S^1 \hookrightarrow S^1 \vee S^1$. Under this isomorphism, which elements of this group correspond to comultiplications on S^1? Which elements are associative comultiplications? Are any of these comultiplications cocommutative? Which element defines π_1? $\square$

Exercise 2.3 Show that the functor π_n preserves products. That is, if X, Y are spaces, then there is a group isomorphism

$$\pi_n(X \times Y) \cong \pi_n X \oplus \pi_n Y, \qquad \forall n \geq 1. \tag{2.3}$$

$\square$

Example 2.1 *For $n \geq 3$ and $k \geq 2$ we define the n-dimensional Moore space of degree k to be the space*

$$P^n(k) = S^{n-1} \cup_k e^n. \tag{2.4}$$

Then $P^n(k)$ is a simply connected CW complex which satisfies

$$\widetilde{H}^r P^n(k) \cong \begin{cases} \mathbb{Z}_k & r = n \\ 0 & \textit{otherwise.} \end{cases} \tag{2.5}$$

In fact this equation uniquely characterises $P^n(k)$ up to homotopy equivalence. That is, any simply connected CW complex with cohomology groups (2.5) is homotopy equivalent $P^n(k)$.

For each $n \geq 3$ the Moore space $P^n(k)$ is a suspension. If $n \geq 4$, then $P^n(k)$ is a double suspension. Thus for a space X, the set

$$\pi_n(X; \mathbb{Z}_k) = [P^n(k), X], \qquad n \geq 3 \tag{2.6}$$

is a group we call the n^{th} homotopy group of X with coefficients in $\mathbb{Z}_k$. If $n \geq 4$ then $\pi_n(X; \mathbb{Z}_k)$ is abelian. We think of these groups as a homotopical analogue of homology with coefficients in $\mathbb{Z}_k$.

Although we can define $P^2(k) = S^1 \cup_k e^2$, this space is not a suspension (assuming $k \neq 0, \pm 1$). In fact it carries no comultiplication (see Example 3.3 below for the reason why). As for larger n, the co-H-structures on $P^n(k)$, $n \geq 4$ are unique. On the other hand it is shown in [2] that the number of comultiplications on $P^3(k)$ is in one-to-one correspondence with the group

$$Ext(\mathbb{Z}_k, \mathbb{Z}_k) \cong \mathbb{Z}_k. \tag{2.7}$$

The paper also addresses the question of which maps $P^m(k) \to P^n(l)$ are co-H-maps. □

3 Recognising co-H-spaces

For the idea to be a good homotopical notion we would expect co-H-structures to be to be stable under homotopy equivalences.

Exercise 3.1 Show that if Y is a co-H-space and $f : X \to Y$ is a map with a left homotopy inverse $g : Y \to X$, then X is a co-H-space. Write down an explicit comultiplication for X. If Y is coassociative or commutative, then does it follow that X is too? What about if f is a homotopy equivalence? □

Thus you have proved:

Proposition 3.1 *Any space homotopy equivalent to a co-H-space is a co-H-space. Any space homotopy equivalent to a cogroup is a cogroup.* ▮

We have defined co-H-spaces and their morphisms. We could conceivably extend this to define a subcategory of $hTop_*$ on the co-H-spaces. By Proposition 3.1 it would even by closed under isomorphism. In fact it would even have coproducts.

Exercise 3.2 Let X, Y be based spaces. Show that the wedge $X \vee Y$ is a co-H-space if and only if both X and Y are co-H-spaces. Show that in this case it is possible to choose co-H-structures on all spaces such that the inclusions $X, Y \hookrightarrow X \vee Y$ and projections $X \vee Y \to X, Y$ are all co-H-maps. □

The presence of a co-H-structure on X has many implications for the structure of X as a space.

Proposition 3.2 *Let X be a CW complex which is a co-H-space. Then $\pi_1 X$ is a free group.*

Exercise 3.3 The condition that X is a CW complex is not strictly necessary. What we need to prove the statement is that we are able to apply the Seifert-van Kampen Theorem to compute

$$\pi_1(X \vee X) \cong \pi_1 X * \pi_1 X \tag{3.1}$$

which will be true in the case that X is CW. Assume that (3.1) holds and study the homomorphism $\pi_1 X \to \pi_1(X \vee X)$ induced by the comultiplication on X to prove Proposition 3.2.[2] $\square$

Example 3.1 *For $n \geq 2$, real projective n-space $\mathbb{R}P^n$ is not a co-H-space.* $\square$

Proposition 3.3 *Let X be a co-H-space. Then all cup products in $\widetilde{H}^* X$ are trivial.*

Exercise 3.4 Prove Proposition 3.3. $\square$

Example 3.2 *For $n \geq 2$, complex projective n-space $\mathbb{C}P^n$ is not a co-H-space.* $\square$

We'll end this be quoting a theorem which is a little harder to prove. You are not required to prove this.

Proposition 3.4 *Let X be a co-H-space and $\varphi : S^n \to X$, $n \geq 1$, a co-H-map. Then $X \cup_\varphi e^{n+1}$ is a co-H-space.* $\blacksquare$

Using this proposition we will be able to construct co-H-space which are not suspensions. For example, there is a co-H-space of the form $S^3 \cup_\varphi e^7$ [1], where φ is an element of order 3 in $\pi_6 S^3$. Of course, before this can be turned into a rigorous example we will have to understand a little bit about the homotopy groups of spheres, and how to detect co-H-maps.

Example 3.3 *The only co-H-maps $S^1 \to S^1$ are the trivial map, the identity and the degree -1 map. For suppose the degree n map $S^1 \to S^1$ were a co-H-map. Then $X = S^1 \cup_n e^2$ would be a co-H-space. But we can use the Seifert-van Kampen theorem to compute that $\pi_1 X \cong \mathbb{Z}_n$, and this group is not free if $n \neq 0, \pm 1$.* $\square$

Example 3.4 *The Hopf map $\eta : S^3 \to \mathbb{C}P^1 \cong S^2$ is not a co-H-map. This is because $\mathbb{C}P^2 \cong S^2 \cup_\eta e^4$, and we have already remarked that $\mathbb{C}P^2$ is not a co-H-space. In fact there is no non-zero multiple of η which is a co-H-map, but we'll need to compute $\pi_3 S^2$ to understand this.* $\square$

[2] Hint: A subgroup of a free group is free.

References

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[3] J.-P. Serre, *Homologie Singulière des Espaces Fibrés*, Applications., Ann. Math. **54** (1951), 425-505.